

A Direct Proof of a Theorem Concerning Singular Hamiltonian Systems

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Abstract

This technical report presents a direct proof of Theorem 1 in [1] and some consequences that also account for (20) in [1]. This direct proof exploits a state space change of basis which replaces the coupled difference equations (10) in [1] with two equivalent difference equations which, instead, are decoupled.

I. INTRODUCTION

Theorem 1 in [1] provides in (19) the set of the admissible solutions (x_k, p_k, u_k) of the singular Hamiltonian system (10) defined on the discrete-time interval $0 \leq k \leq k_f - 1$. The proof therein presented is twofold: sufficiency is shown by direct replacement of (19) in (10); necessity relies on maximality of the involved structural invariant subspaces, as it is deducible from Properties 1 and 2. In the following, it will be shown that a direct proof, which does not distinguish between the *if* and the *only-if* part, but extensively uses relations pointed out in [1], is also feasible. The main point of the direct proof is replacing the coupled difference equations (10) in [1] with two decoupled difference equations by means of a suitable state space basis transformation. The direct proof herein presented can also be used to prove (20) in [1], that expresses the set of the admissible solutions (x_k, p_k) of the same Hamiltonian system in the extended time interval $0 \leq k \leq k_f$.

II. DIRECT PROOF OF RELATION (20) AND THEOREM 1 IN [1]

The direct proof is based on the following lemma.

Lemma 1: The problem of finding the sequences x_k , p_k , and u_k , with $0 \leq k \leq k_f - 1$, that solve the equations (10) of [1], or, equivalently,

$$x_{k+1} = A x_k + B u_k, \quad (1)$$

$$-A^\top p_{k+1} = Q x_k - p_k + S u_k, \quad (2)$$

$$-B^\top p_{k+1} = S^\top x_k + R u_k, \quad (3)$$

with $0 \leq k \leq k_f - 1$, can be reduced to that of finding the sequences v_k and w_k that solve the pair of decoupled difference equations

$$v_{k+1} = A_+ v_k, \quad (4)$$

$$A_+^\top w_{k+1} = w_k, \quad (5)$$

with $0 \leq k \leq k_f - 1$, where A_+ is defined by (14) in [1], provided that the following correspondences are set up

$$x_k = v_k + W w_k, \quad (6)$$

$$p_k = P_+ v_k + (-I + P_+ W) w_k, \quad (7)$$

$$u_k = -K_+ v_k + \bar{K}_+ w_{k+1}, \quad (8)$$

where P_+ is the positive semidefinite symmetric solution of (11)–(12) in [1], W is the solution of the symmetric discrete Lyapunov equation (15), K_+ , and \bar{K}_+ are defined by (13) and (17).

Proof: First, the following relation will be shown:

$$-W + B\bar{K}_+ = -A W A_+^\top. \quad (9)$$

Use of (17) in [1] yields the identity

$$-W + B\bar{K}_+ = -W + B(R + B^\top P_+ B)^{-1}(B^\top - B^\top P_+ A W A_+^\top - S^\top W A_+^\top) =$$

and, by applying distributivity of the product with respect to the sum,

$$\begin{aligned} &= -W + B(R + B^\top P_+ B)^{-1}B^\top - B(R + B^\top P_+ B)^{-1}B^\top P_+ A W A_+^\top \\ &\quad - B(R + B^\top P_+ B)^{-1}S^\top W A_+^\top = \end{aligned}$$

and, by collecting $W A_+^\top$ in the last two terms,

$$= -W + B(R + B^\top P_+ B)^{-1}B^\top - B(R + B^\top P_+ B)^{-1}(B^\top P_+ A + S^\top) W A_+^\top =$$

and, by the definition (13) of K_+ in [1], and summing and subtracting the term $A W A_+^\top$

$$= -W + B(R + B^\top P_+ B)^{-1}B^\top - B K_+ W A_+^\top + A W A_+^\top - A W A_+^\top =$$

and, by reordering,

$$= (A - B K_+) W A_+^\top - W + B(R + B^\top P_+ B)^{-1}B^\top - A W A_+^\top =$$

and, by using (14) in [1],

$$= A_+ W A_+^\top - W + B(R + B^\top P_+ B)^{-1}B^\top - A W A_+^\top =$$

and, eventually, tacking (15) in [1] into account,

$$= -A W A_+^\top.$$

Thus, (9) is proven. Now we are ready to obtain the difference equation in the unknowns v_k and w_k . By using (6) and (8) in (1), it follows that:

$$v_{k+1} + W w_{k+1} = A v_k + A W w_k - B K_+ v_k + B \bar{K}_+ w_{k+1},$$

or also

$$v_{k+1} = (A - B K_+) v_k + (-W + B \bar{K}_+) w_{k+1} + A W w_k,$$

or, by the definition (14) in [1],

$$v_{k+1} = A_+ v_k + (-W + B \bar{K}_+) w_{k+1} + A W w_k, \quad (10)$$

or, equivalently because of (9),

$$v_{k+1} = A_+ v_k - A W A_+^\top w_{k+1} + A W w_k. \quad (11)$$

Similarly, by using (6)–(8) in (2), the following is obtained:

$$\begin{aligned} & -A^\top(P_+v_{k+1} + (P_+W - I)w_{k+1}) = \\ & = Q(v_k + Ww_k) - (P_+v_k - (P_+W - I)w_k) + S(-K_+v_k + \bar{K}_+w_{k+1}), \end{aligned}$$

or

$$\begin{aligned} & -A^\top P_+v_{k+1} - A^\top(P_+W - I)w_{k+1} = \\ & = Qv_k + QWw_k - P_+v_k - (P_+W - I)w_k - SK_+v_k + S\bar{K}_+w_{k+1}. \end{aligned}$$

By the identity $-A^\top(P_+W - I) = QWA_+^\top - (P_+W - I)A_+^\top + S\bar{K}_+$ (see the proof of Property 2 in [1] – second row block), the following holds:

$$\begin{aligned} & -A^\top P_+v_{k+1} + (QWA_+^\top - (P_+W - I)A_+^\top + S\bar{K}_+)w_{k+1} = \\ & = Qv_k + QWw_k - P_+v_k - (P_+W - I)w_k - SK_+v_k + S\bar{K}_+w_{k+1}, \end{aligned}$$

and, by doing away with the terms $S\bar{K}_+w_{k+1}$ at the right of both members,

$$\begin{aligned} & -A^\top P_+v_{k+1} + (QW - (P_+W - I))A_+^\top w_{k+1} = \\ & = (Q - P_+ - SK_+)v_k + (QW - (P_+W - I))w_k. \end{aligned}$$

Recall the identity $Q - P_+ - SK_+ = -A^\top P_+A_+$ (see the proof of Property 1 in [1] – second row block), the following is obtained:

$$-A^\top P_+v_{k+1} + (QW - (P_+W - I))A_+^\top w_{k+1} = -A^\top P_+A_+v_k + (QW - (P_+W - I))w_k. \quad (12)$$

Let us multiply both members of (11) by $A^\top P_+$, thus obtaining

$$A^\top P_+v_{k+1} = A^\top P_+A_+v_k - A^\top P_+AWA_+^\top w_{k+1} + A^\top P_+AWw_k, \quad (13)$$

and, by summing both members of (12) and (13), it follows that

$$\begin{aligned} & (QW - (P_+W - I))A_+^\top w_{k+1} = \\ & = (QW - (P_+W - I))w_k - A^\top P_+AWA_+^\top w_{k+1} + A^\top P_+AWw_k. \end{aligned}$$

By collecting w_{k+1} on the left and w_k on the right, it follows that

$$(QW - (P_+W - I) + A^\top P_+AW)A_+^\top w_{k+1} = (QW - (P_+W - I) + A^\top P_+AW)w_k,$$

or $A_+^\top w_{k+1} = w_k$, that is (5). Taking into account this latter equation in (11) one gets

$$v_{k+1} = A_+v_k - AWw_k + AWw_k,$$

or $v_{k+1} = A_+v_k$, that is (4). ■

Now we are ready to conclude the direct proof of both (20) and (19) in [1]. Refer to the pair of decoupled difference equations (4), (5), defined in the time interval $0 \leq k \leq k_f - 1$. Their solutions can be expressed as

$$\begin{aligned} v_k &= A_+^k \alpha, \\ w_k &= (A_+^\top)^{k_f - k} \beta, \end{aligned} \quad 0 \leq k \leq k_f, \quad (14)$$

where $\alpha, \beta \in \mathbb{R}^n$ are parameters. Substitution of (14) in (6), (7) yields

$$\begin{aligned} x_k &= A_+^k \alpha + W(A_+^\top)^{k_f - k} \beta, \\ p_k &= P_+A_+^k \alpha + (P_+W - I)(A_+^\top)^{k_f - k} \beta, \end{aligned} \quad 0 \leq k \leq k_f,$$

that, re-written in compact notation as

$$\begin{bmatrix} x_k \\ p_k \end{bmatrix} = \begin{bmatrix} I \\ P_+ \end{bmatrix} A_+^k \alpha + \begin{bmatrix} W \\ P_+ W - I \end{bmatrix} (A_+^\top)^{k_f-k} \beta, \quad 0 \leq k \leq k_f,$$

coincides with equation (20) in [1].

To prove equation (19) in [1], let us substitute (5), i.e.,

$$w_k = A_+^\top w_{k+1}, \quad 0 \leq k \leq k_f - 1,$$

in (6), (7), thus obtaining

$$\begin{aligned} x_k &= v_k + W A_+^\top w_{k+1}, \\ p_k &= (P_+ W - I) A_+^\top w_{k+1}, \end{aligned} \quad 0 \leq k \leq k_f - 1. \quad (15)$$

Using (14) in (15) yields

$$\begin{aligned} x_k &= A_+^k \alpha + W A_+^\top (A_+^\top)^{k_f-k-1} \beta, \\ p_k &= P_+ A_+^k \alpha + (P_+ W - I) A_+^\top (A_+^\top)^{k_f-k-1} \beta, \end{aligned} \quad 0 \leq k \leq k_f - 1, \quad (16)$$

while using (14) in (8) provides

$$u_k = -K_+ A_+^k \alpha + \bar{K}_+ (A_+^\top)^{k_f-k-1} \beta, \quad 0 \leq k \leq k_f - 1. \quad (17)$$

Equations (16), (17) can be re-written in compact form as

$$\begin{bmatrix} x_k \\ p_k \\ u_k \end{bmatrix} = \begin{bmatrix} I \\ P_+ \\ -K_+ \end{bmatrix} A_+^k \alpha + \begin{bmatrix} W A_+^\top \\ (P_+ W - I) A_+^\top \\ \bar{K}_+ \end{bmatrix} (A_+^\top)^{k_f-k-1} \beta, \quad 0 \leq k \leq k_f - 1,$$

that coincides with (19) in [1]. Thus, Theorem 1 in [1] has been directly proven by using the correspondences stated in Lemma 1.

REFERENCES

- [1] E. Zattoni, "Structural invariant subspaces of singular Hamiltonian systems and nonrecursive solutions of finite-horizon optimal control problems," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1279–1284, June 2008.